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Functions of the free field: examples for essentially local non-localizable fields

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Abstract. By extensive use of methods developed by Rieckers it is shown that power series of the free field, while not strictly localizable in general, are essentially local with respect to all those Gel'fand spaces S^{α} on which they can be defined at all.

1. Introduction

In a previous paper (Bümmerstede and Lücke 1974) the notion of essential locality was introduced as a natural generalization of locality for non-localizable fields, ie for fields defined on test spaces not containing functions with compact support in x space. It was claimed that examples for non-localizable fields fulfilling essential locality, which are not just restrictions of localizable fields to non-localizable test spaces, do in fact exist† and are provided by functions of the free field as treated by Rieckers (1971). The purpose of the present paper is to prove this statement.

As in the preceding paper (Bümmerstede and Lücke 1974) we shall restrict our discussion to a single type of neutral scalar particle with mass $m > 0$, described by the hermitian Wightman field $A(x)$ on the Gel'fand space (Gel'fand and Schilow 1962, Rieckers 1971) $S^{s}(R^{4}) = S^{s,s,s,s}$, $s \geqslant 0$. This means the usual Wightman axioms (Streater and Wightman 1964) are required to hold with just two modifications:

(i) Locality is to be replaced by essential locality (reviewed below).

(ii) The Schwartz space $\mathscr{S}(R^{4})$ of tempered functions is to be replaced by the Gel'fand space $S^{s}(R^{4})$, $s \geqslant 0$.

Let D be the common invariant dense domain of the field operators (smeared fields) in the Hilbert space of states $\mathscr{H}$. Then (Bümmerstede and Lücke 1974, Lemma 2) the vacuum expectation values

$$\langle \Phi | [A(x), A(y)] | \Psi \rangle, \; \Phi, \Psi \in D \tag{1.1}$$

primarily defined as bilinear functionals over $S^{s}(R^{4}) \times S^{s}(R^{4})$, have a unique extension to continuous linear functionals (generalized functions) over $S^{s}(R^{8}) = S^{s}(R^{4}) \hat{\otimes} S^{s}(R^{4})$. This allowed the following two definitions (Bümmerstede and Lücke 1974):

A subset S of $S^{s}(R^{8})$ is called *locally bounded* on‡

$$V_{8} \equiv \{\hat{x} = (x_{1}, x_{2}) \in R^{8} : (x_{1} - x_{2})^{2} \geqslant 0\}$$

† Of course, we do not know *explicit* examples with non-trivial scattering.

‡ We use standard notation as in the preceding paper (Bümmerstede and Lücke 1974).

if there are positive constants e, A such that

$$\sup_{\varphi\in S}\ \sup_{\hat{x}\in U_e(V_8)}\ \sup_{\hat{\alpha}\in Z_+^8} A^{-|\hat{\alpha}|}\hat{\alpha}^{-s\hat{\alpha}}\|\hat{x}\|^N|\varphi^{(\hat{\alpha})}(\hat{x})| < \infty$$

holds for every non-negative integer N. The field $A(x)$ on $S^s(R^4)$ is called *essentially local* if

$$\sup_{\varphi\in S}\left|\int \langle\Phi|[A(x_1), A(x_2)]|\Psi\rangle\varphi(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\right| < \infty$$

for arbitrary Φ, $\Psi \in D$ and for every subset S of $S^s(R^4)$ which is locally bounded on V_8.

2. Functions of the free field

The convergence of power series

$$A(\varphi) \equiv \sum_{r=0}^{\infty}\frac{d_r}{r!}:A_0^r:(\varphi) \tag{2.1}$$

of the free hermitian Wightman field A_0, mass $m > 0$, was extensively analysed by Rieckers (1971). By inspection of his methods and results we get almost complete information about the solution of our problem. Therefore, let us briefly review Rieckers' results as far as they are relevant for our purposes.

Let $\mathscr{H}$ be the Fock space of the free field, $\|\,.\,\|$ its norm, and Ω the Fock vacuum. Consider $s \in [0, 1)$ and, in case $s > 0$, suppose

$$\sup_{r\in Z_+}\frac{(\mathrm{d}_r)^2}{r!}\exp(-r^b) < \infty \tag{2.2}$$

for some $b \in (1, 1/s)$. Then, for φ, $\psi_j \in S^s(R^4)$, Rieckers proved that the strong limits

$$\mathrm{s}-\lim_{N\to\infty}\prod_{j=1}^{k}:A_0^{r_j}:(\psi_j)\sum_{r=0}^{N}\frac{\mathrm{d}_r}{r!}:A_0^r:(\varphi)\Psi, \qquad k = 0, 1, \ldots \tag{2.3}$$

exist in $\mathscr{H}$, where the vector $\Psi \in \mathscr{H}$ may be recursively taken of the form

$$\prod_{k=1}^{n} A(\varphi_k)\Omega; \qquad n = 0, 1, 2, \ldots \tag{2.4}$$

with arbitrary $\varphi_1, \ldots, \varphi_n \in S^s(R^4)$. Hence, if we define D to be the linear hull of the set of all vectors of the form (2.4) and if $d_1 \neq d_0 = 0$, (2.1) has a natural definition as a scalar Wightman field $A(x)$ on $S^s(R^4)$ with dense invariant domain D in $\mathscr{H}$. We postulate $d_r = d_r^*$, hence $A(x)$ is hermitian.

Moreover, suppose

$$\sup_{r\in Z_+}\frac{(d_r)^2}{r!}\exp(-r^{1/s_0}) = \infty \tag{2.5}$$

for some $s_0 \in (s, 1)$. Then the two-point function

$$\langle\Omega|A(x_1)A(x_2)|\Omega\rangle$$

cannot be extended to a continuous linear functional over $S^{s_0}(R^8)$. As a consequence, the field is necessarily non-localizable. We rephrase Rieckers' argument in order to show that also the vacuum expectation value of the field commutator cannot be defined on $S^{s_0}(R^8)$:

Let $\{\tilde{g}_k\}_{k\in Z_+} \subset \mathscr{D}(R^8)$ be a partition of unity. So all the $\tilde{g}_k$ have non-negative values and for arbitrary $\tilde{\varphi} \in \widetilde{S^{s_0}}(R^8)$ the series

$$\sum_{k=0}^{\infty} \tilde{\psi}_k = \tilde{\varphi}, \qquad \tilde{\psi}_k \equiv \tilde{g}_k \tilde{\psi}$$

converges in the topology of $\widetilde{S^{s_0}}(R^8)$. The $\tilde{g}_k$ may be chosen so that

$$\tilde{g}_k(p_1, p_2) = \tilde{g}_k(p_2, p_1)$$

holds for all $p_1, p_2 \in R^4$. Moreover, let us choose some $\tilde{h}_\pm \in \widetilde{S^{s_0}}(R^4)$ fulfilling

$$\pm\tilde{h}_\pm(-p) = \tilde{h}_\pm(p) = \exp\{-\tfrac{1}{2}a[\tfrac{1}{2}p^0 + \tfrac{1}{2}(\boldsymbol{p}^2+m^2)^{1/2}]^{1/s_0}\}$$

for given $a > 0$ and for all $p \in R^4$ with $p^0 \geqslant \frac{1}{2}m$, and define

$$\tilde{\varphi}_a(p_1, p_2) \equiv \tilde{h}_+(p_1)\tilde{h}_-(p_2).$$

Now, if the vacuum expectation value of the field commutator could be extended to a generalized function over $S^{s_0}(R^8)$, we had $[p_l^0 = (\boldsymbol{p}_l^2+m^2)^{1/2}]$:

$$\int dx_1\, dx_2 \langle\Omega|[A(x_1), A(x_2)]|\Omega\rangle \varphi_a(x_1, x_2)$$

$$= \sum_{k=0}^{\infty} \int dx_1\, dx_2 \langle\Omega|[A(x_1), A(x_2)]|\Omega\rangle \psi_k(x_1, x_2)$$

$$= 2\sum_{k=0}^{\infty}\sum_{r=0}^{\infty} \frac{(d_r)^2}{r!}(2\pi)^{-3(r-1)} \int \prod_{l=1}^{r} \frac{d\boldsymbol{p}_l}{2p_l^0}(\tilde{g}_k\tilde{\varphi}_a)\left(-\sum_{l=1}^{r} p_l, +\sum_{l=1}^{r} p_l\right).$$

Since the latter expression is a sum over non-negative terms, we could interchange summations and obtain:

$$\int dx_1\, dx_2 \langle\Omega|[A(x_1), A(x_2)]|\Omega\rangle \varphi_a(x_1, x_2)$$

$$= 2\sum_{r=0}^{\infty} \frac{(d_r)^2}{r!}(2\pi)^{-3(r-1)} \int \prod_{l=1}^{r} \frac{d\boldsymbol{p}_l}{2p_l^0}\tilde{\varphi}_a\left(-\sum_{l=1}^{r} p_l, +\sum_{l=1}^{r} p_l\right).$$

However, for sufficiently small a the sum on the RHS is divergent, since we have (2.5) and since (Rieckers 1971)

$$\int \prod_{l=1}^{r} \frac{d\boldsymbol{p}_l}{2p_l^0}\tilde{\varphi}_a\left(-\sum_{l=1}^{r} p_l, +\sum_{l=1}^{r} p_l\right) \geqslant C_1[\overline{M}(a)]^r \exp(-r^{1/s_0})$$

holds with $\overline{M}(a) \to \infty$ for $a \to 0$. We conclude: At least for $\Psi = \Phi = \Omega$ (1.1) cannot be extended to a generalized function over $S^{s_0}(R^8)$, therefore $A(x)$ is necessarily non-localizable.

So we are left to prove essential locality of the field (2.1) under the above conditions.

3. Fundamental local estimates

For a successful analysis of the convergence of (2.1) *global* estimates on distributions of the form

$$[i\Delta_m^+(x_1-x_2)]^{r_{1,2}}[i\Delta_m^+(x_1-x_3)]^{r_{1,3}} \ldots [i\Delta_m^+(x_{n-1}-x_n)]^{r_{n-1,n}}$$

turned out to be sufficient (Rieckers 1971). In order to prove essential locality of $A(x)$ we need *local* estimates on distributions of the form

$$\Delta^{r_{j,k}}_{n,n'}(x_1,\ldots,x_n) \equiv \{[\mathrm{i}\Delta_m^+(x_{n'}-x_{n'+1})]^{r_{n',n'+1}}-[\mathrm{i}\Delta_m^+(x_{n'+1}-x_{n'})]^{r_{n',n'+1}}\} \times \prod_{\substack{0<j<k\leqslant n\\(j,k)\neq(n',n'+1)}} [\mathrm{i}\Delta_m^+(x_j-x_k)]^{r_{j,k}}$$

with respect to the region $x_{n'}-x_{n'+1}\in\bar{V}$. More precisely, given positive constants e, A, B, we have to find an asymptotically sufficiently small functional $M(r_{j,k})$ fulfilling

$$\left|\int \mathrm{d}\hat{x}\Delta^{r_{j,k}}_{n,n'}(\hat{x})\varphi(\hat{x})\right| < M(r_{j,k}) \tag{3.1}$$

for all $\varphi\in S^s(R^{4n})$ with

$$\sup_{\substack{\hat{x}\in R^{4n}\\(x_{n'},x_{n'+1})\in U_e(V_8)}} \sup_{\hat{\alpha}\in Z_+^{4n}} A^{-|\hat{\alpha}|}\hat{\alpha}^{-s\hat{\alpha}}(1+\|\hat{x}\|)^{4(n+1)}|\varphi^{(\hat{\alpha})}(x)| < B. \tag{3.2}$$

While the $r_{j,k}$ may be arbitrary non-negative integers for $(j,k)\neq(n',n'+1)$, $r_{n',n'+1}$ is an arbitrary *positive* integer. In order to get such estimates, we modify Rieckers' technique in the following way:

Using difference variables

$$\xi_j \equiv x_j-x_{j+1}\,;\, j=1,\ldots,n \qquad (x_{n+1}\equiv 0),$$

define

$$h(\xi_1,\ldots,\xi_{n-1}) \equiv \int \mathrm{d}\xi_n\varphi\left(\sum_{j=1}^{n}\xi_j,\ldots,\sum_{j=n}^{n}\xi_j\right)$$

and take a suitable multiplier $g\in\mathcal{O}_M(R^4)$ with $g(x)=1$ in some neighbourhood of $\bar{V}$. Then, recalling that odd Lorentz invariant Schwartz distributions have supports contained in $\bar{V}$, we have

$$\int \mathrm{d}\hat{x}\Delta^{r_{j,k}}_{n,n'}(\hat{x})\varphi(\hat{x}) = C^R\int \prod_{\substack{0<j<k\leqslant n\\(j,k)\neq(n',n'+1)}} \prod_{l=1}^{r_{j,k}} \frac{\mathrm{d}\boldsymbol{p}_{j,k,l}}{2p^0_{j,k,l}} \int \prod_{l=1}^{r_{n',n'+1}} \frac{\mathrm{d}\boldsymbol{p}_{n',n'+1,l}}{2p^0_{n',n'+1,l}}$$
$$\times(q_nq_n)^{-K}\int \mathrm{d}q'[\tilde{g}(-q_n-q')-\tilde{g}(+q_n-q')](q'q')^{+K}$$
$$\times\tilde{h}(-q_1,\ldots,-q_{n'}+q',\ldots,-q_{n-1})$$

for every non-negative integer K, where

$$R\equiv\sum_{0<j<k\leqslant n} r_{j,k}, \qquad q_{n'}\equiv\sum_{\substack{0<j\leqslant n'<k\leqslant n\\(j,k)\neq(n',n'+1)}}\sum_{l=1}^{r_{j,k}} p_{j,k,l'}$$

$$q_{j'}\equiv\sum_{j=1}^{j'}\sum_{k=j'+1}^{n}\sum_{l=1}^{r_{j,k}} p_{j,k,l} \qquad \text{for } n'\neq j'<n,\; q_n\equiv\sum_{l=1}^{r_{n',n'+1}} p_{n',n'+1,l}.$$

Consequently†,

$$\left|\int \mathrm{d}\hat{x}\Delta^{r_{j,k}}_{n,n'}(\hat{x})\varphi(\hat{x})\right| \leqslant C^{R}(mR_n)^{-2K}J(\varphi, K; L_1, \ldots, L_n)\prod_{j=1}^{n} I(R_j, L_j)$$

holds for $L_j > 2R_j$ with

$$R_j \equiv \sum_{k=j+1}^{n} r_{j,k} \qquad \text{for } n' \neq j < n, \qquad R_{n'} \equiv \sum_{k=n'+2}^{n} r_{n',k}$$

$$R_n \equiv r_{n',n'+1}, \qquad R = \sum_{j=1}^{n} R_j$$

$$J(\varphi, K; L_1, \ldots, L_n) \equiv \max_{p_{j,k,l}\in R^3}\left|\prod_{j=1}^{n}(q_j^0)^{L_j}\int \mathrm{d}q'[\tilde{g}(-q_n-q')-\tilde{g}(+q_n-q')](q'q')^K \times \tilde{h}(-q_1, \ldots, -q_{n'}+q', \ldots, -q_{n-1})\right|$$

and (Rieckers 1971):

$$\begin{aligned} I(r, l) &\equiv \int \prod_{j=1}^{r}\frac{\mathrm{d}\boldsymbol{p}_j}{2p_j^0}\left(\sum_{j=1}^{r} p_j^0\right)^{-l} \qquad (\equiv 1 \quad \text{for } r = 0) \\ &< C^r m^{-l}\frac{r^{r-l}}{(r-1)!(l-2r)} \qquad \text{for } l > 2r > 0. \end{aligned} \tag{3.4}$$

The four-momenta p_j, $p_{j,k,l}$ are always on-shell, ie:

$$p_j^0 \equiv (\boldsymbol{p}_j^2+m^2)^{1/2}, \qquad p_{j,k,l}^0 \equiv (\boldsymbol{p}_{j,k,l}^2+m^2)^{1/2}.$$

Now the problem is to suitably choose g and the non-negative integers $K = K(R_n)$, $L_j = L_j(R_j)$ in order to make the RHS of (3.3) asymptotically ($r_{j,k} \to \infty$) sufficiently small.

Let us first estimate $J(\varphi, K; L_1, \ldots, L_n)$. By

$$(q'q') = (q_{n'}-q')(q_{n'}-q') - 2q_{n'}(q_{n'}-q') + q_{n'}q_{n'}$$

we may write

$$(q'q')^K = \sum_{\substack{\alpha,\beta\in Z_+^4 \\ |\alpha|+|\beta|=2K}} b_{\alpha,\beta}q_{n'}^{\alpha}(q_{n'}-q')^{\beta}$$

with

$$|b_{\alpha,\beta}| < C^K.$$

Therefore, since

$$|q_{n'}^{\alpha}| < \left(\sum_{n'\neq j<n} q_j^0\right)^{|\alpha|}$$

we have

$$J(\varphi, K; L_1, \ldots, L_n) \geqslant 2C^{K+L}\max_{\substack{\beta\in Z_+^4 \\ |\beta|\leqslant 2K}}\max_{q'_j\in R^4}\left|\int \mathrm{d}q'(q_n'^0)^{L_n}\tilde{g}(q'_n-q')(q'_{n'}-q')^{\beta} \times \left(\sum_{j=1}^{n-1} q_j'^0\right)^{L+2K-|\beta|}\tilde{h}(-q'_1, \ldots, -q'_{n'}+q', \ldots, -q'_{n-1})\right|$$

† We adopt the convention that constants C, $C_1, \ldots$ may have different values in different lines!

where

$$L \equiv \sum_{j=1}^{n-1} L_j.$$

In configuration space we thus obtain

$$J(\varphi, K; L_1, \dots, L_n)$$
$$\leqslant C_1 C^{K+L} \max_{\substack{\alpha_j \in Z_+^4 \\ \sum_{k=1}^{n-1} |\alpha_k| = L+2K}} \max_{\substack{\xi_j \in R^4 \\ \xi_{n'} \in U_e(V)}} [1 + \|(\xi_1, \dots, \xi_{n-1})\|]^{4n}$$
$$\times \left| \left(\frac{\partial}{\partial \xi_{n'}^0} \right)^{L_n} \left[g(\xi_{n'}) \prod_{j=1}^{n-1} D_{\xi_j}^{\alpha_j} h(\xi_1, \dots, \xi_{n-1}) \right] \right|$$

and hence by (3.2) for fixed A, B:

$$J(\varphi, K; L_1, \dots, L_n)$$
$$\leqslant C_1 C^{K+L+L_n} (L + L_n + 2K)^{s(L+L_n+2K)} \max_{\substack{l \in Z_+ \\ l \leqslant L_n}} \max_{x \in R^4} \left| \left(\frac{\partial}{\partial x^0} \right)^l g(x) \right|.$$

We may choose a non-negative function $\delta_e \in S^2(R^4)$ with

$$\int dx \delta_e(x) = 1$$

and (Gel'fand and Schilow 1962):

$$\delta_e(x) = 0 \qquad \text{for } \|x\| > e/4.$$

Next define

$$g(x) \equiv \int_{U_{e/2}(V)} dx' \delta_e(x - x').$$

Then we have $g \in \mathscr{O}_M(R^4)$ and

$$g(x) = \begin{cases} 1 & \text{if } x \in U_{e/4}(V) \\ 0 & \text{if } x \notin U_e(V). \end{cases}$$

Moreover,

$$\sup_{x \in R^4} |\partial_0^l g(x)| < C_1 \max_{x \in R^4} |\partial_0^l \delta_e(x)|$$
$$< C_2 C^l l^{2l}$$

holds for all $l \in Z_+$. With this choice for g we finally have

$$J(\varphi, K; L_1, \dots, L_n) < C_1 C^{K+L+L_n} L_n^{3L_n} (L + 2K)^{s(L+2K)}.$$

Thus, putting

$$L_n = 2R_n + 1, \qquad K_n = K$$
$$K_j = L_j/2 \qquad \text{for } j < n$$

(3.3) and (3.4) yield

$$\left| \int \mathrm{d}\hat{x} \Delta_{n,n'}^{r_{j,k}}(\hat{x}) \varphi(\hat{x}) \right| < C^R R_n^{4R_n} \prod_{j=1}^{n} C^{K_j} K_j^{2sK_j} R_j^{-2K_j} \tag{3.5}$$

for all integer $K_j > R_j$ and for all $\varphi \in S^s(R^{4n})$ fulfilling (3.2).

4. Proof of essential locality

Now we are well prepared for the proof of essential locality of the fields $A(x)$ defined in § 2.

For $0 < n' < n$ define the following generalized function (see Bümmerstede and Lücke 1974) over $S^s(R^{4n})$:

$$W_{n,n'}(x_1, \ldots, x_n) \equiv \left\langle \Omega \middle| \prod_{j=1}^{n'-1} A(x_j)[A(x_{n'}), A(x_{n'+1})] \prod_{k=n'+2}^{n} A(x_k) \middle| \Omega \right\rangle .$$

Then the field $A(x)$ is essentially local if

$$\sup_{\varphi \in S} \left| \int \mathrm{d}\hat{x} W_{n,n'}(\hat{x}) \varphi(\hat{x}) \right| < \infty$$

holds for every subset S of $S^s(R^{4n})$ that is *locally bounded* on

$$\{\hat{x} = (x_1, \ldots, x_n) : (x_{n'}, x_{n'+1}) \in V_8\} ;$$

ie for which there are constants e, A and B such that (3.2) holds for all $\varphi \in S$.

Note that by (2.3), (2.4), and Wick's theorem we have:

$$\begin{aligned} W_{n,n'}(x_1, \ldots, x_n) &= \lim_{N_{n'+1} \to \infty} \ldots \lim_{N_1 \to \infty} \lim_{N_{n'+2} \to \infty} \ldots \lim_{N_n \to \infty} \sum_{r_1=1}^{N_1} \ldots \sum_{r_n=1}^{N_n} \prod_{l=1}^{n} \frac{d_{r_l}}{r_l!} \\ &\times \left\langle \Omega \middle| \prod_{j=1}^{n'-1} :A_0^{r_j}:(x_j)[:A_0^{r_{n'}}:(x_{n'}), :A_0^{r_{n'+1}}:(x_{n'})] \prod_{k=n'+2}^{n} :A_0^{r_k}:(x_k) \middle| \Omega \right\rangle \\ &= \lim_{N_{n'+1} \to \infty} \ldots \lim_{N_1 \to \infty} \lim_{N_{n'+2} \to \infty} \ldots \lim_{N_n \to \infty} \sum_{r_1=1}^{N_1} \ldots \sum_{r_n=1}^{N_n} \\ &\times \sum_{r_{j,k} \in R(r_1, \ldots, r_n)} \frac{d_{r_1} \ldots d_{r_n}}{r_{1,2}! r_{1,3}! \ldots r_{n-1,n}!} \Delta_{n,n'}^{r_{j,k}}(x_1, \ldots, x_n) \end{aligned}$$

where $R(r_1, \ldots, r_n)$ denotes the set of all Z_+-valued applications $r_{j,k}$, defined for $(j,k) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ and fulfilling

$$r_{j,k} = r_{k,j}, r_{k,k} = 0, \sum_{k=1}^{n} r_{j,k} = r_j .$$

Thus, recalling the local estimates (3.5), it is quite sufficient to prove convergence of the series

$$\sum_{r_1, \ldots, r_n = 1}^{\infty} \sum_{r_{j,k} \in R(r_1, \ldots, r_n)} \frac{d_{r_1} \ldots d_{r_n}}{r_{1,2}! r_{1,3}! \ldots r_{n-1,n}!} C^R R_n^{4R_n} \prod_{l=1}^{n} [CK(R_l)^{2s} R_l^{-2}]^{K(R_l)} \tag{4.1}$$

for at least one integer valued function $K(r) > r$. For $s = 0$ we only have to choose $K(r)$

of sufficiently rapid increase in order to guarantee convergence of (4.1). For $s > 0$, on the other hand, (2.2) holds by assumption and therefore a simple estimate (compare Rieckers 1971, Lemma 2) gives

$$|d_{r_1} \dots d_{r_n}| \geqslant \prod_{0<j<k\leqslant n} r_{j,k}! C_1^{r_{j,k}^b}.$$

We choose $K(r)$ such that

$$r^b < K(r) \leqslant 1 + r^b$$

holds for all $r \in Z_+$. This ensures

$$C^R R_n^{4R_n} \prod_{l=1}^{n} [CK(R_l)^{2s} R_l^{-2}]^{K(R_l)} < C_2 \prod_{0<j<k\leqslant n} (r_{j,k})^{-\delta r_{j,k}^b}$$

with $\delta \equiv 1 - sb > 0$, so (4.1) may be majorized by

$$C_2 \prod_{0<j<k\leqslant n} \sum_{r_{j,k}=0}^{\infty} C_1^{r_{j,k}^b} (r_{j,k})^{-\delta r_{j,k}^b}$$

which is obviously convergent. Hence $A(x)$ is essentially local in both the cases $s = 0$ and $s > 0$.

5. Conclusions

We have seen that an arbitrary power series of the free field $A_0(x)$

$$A(x) = \sum_{r=0}^{\infty} \frac{d_r}{r!} :A_0^r:(x)$$

is essentially local with respect to every $S^s(R^4)$, $0 \leqslant s < 1$, on which it can be defined. The proof of essential locality was based on essentially the same estimates necessary for just the definition of $A(x)$. Hopefully, this is a general feature of essential locality.

While localizability of the vacuum expectation values of the field $A(x)$ requires certain restrictions on the growth of the d_r for $r \to \infty$, there are no restrictions whatsoever for the field to be defined (and hence essentially local) on $S^0(R^4)$. We conclude that essential locality is more general than ordinary locality.

Of course, functions of the free field are trivial when regarded as 'interacting' fields. For a natural definition of non-localizable superpropagators, however, essential locality of power series of the free field turns out to be a very useful property (J Bümmerstede, thesis in preparation).

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